Introduction to 3-Space

GOAL: Systematically understand Euclidean 3-space so we can begin to do multivariable calculus.

Our world is three dimensional, and a major goal of Calculus III is to understand the geometry of this 3D space so we can understand its calculus.

We accomplish this by bootstrapping our knowledge of two-dimensional space (i.e. of plane geometry).

I. Structure

Every point in 3-space is a triple of real numbers.

3-space → $\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ ← set notation

Notation: $\mathbb{R}^3$ ⇝ Euclidean 3-space

$\mathbb{R}$ ⇝ the real line / real numbers.

$\in$ ⇝ symbol meaning "is an element of" from set notation ...

Hence we have a good formal model for our three dimensional space ...

II. Distances

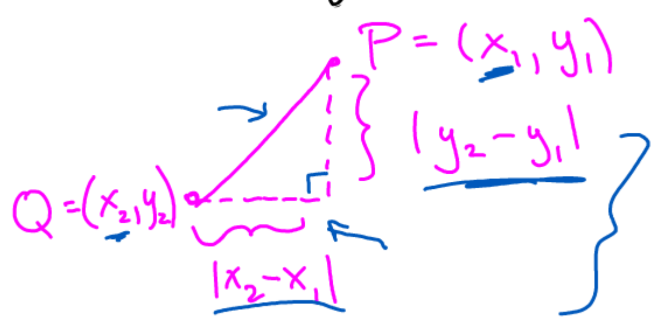
The first step to understanding the geometry of $\mathbb{R}^3$ is understanding distances in $\mathbb{R}^3$...

Q: How far apart are two given points in $\mathbb{R}^3$? How do we figure it out?

Ans: Bootstrap the Pythagorean Theorem...

Given two points in a plane, we can draw a right triangle so the two points form the hypoteneuse.

The legs of this triangle have length the absolute difference of the components.



Hence we can use the Pythagorean theorem to compute the distance between these points:

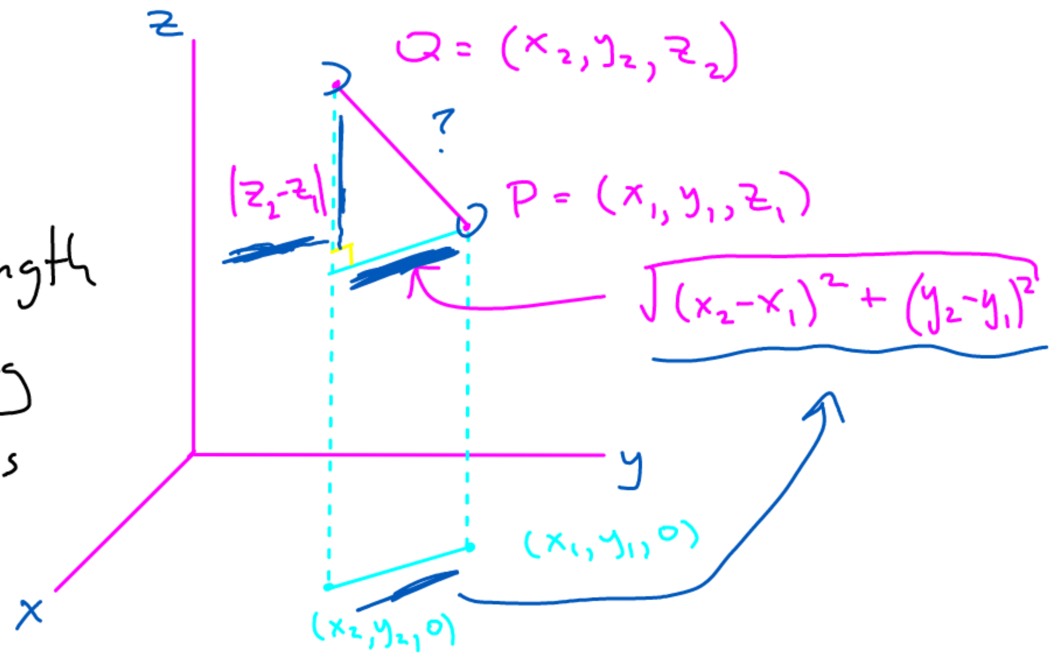
$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

But that was inherently two dimensional.

How do we make the argument work in 3D?

Ans: We'll force the argument into a plane first!

Hence the length of the connecting line segment is precisely...



$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

We just proved our first theorem ☺

Prop (DISTANCE FORMULA): Two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ are at distance $d(P, Q) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

NB: A similar formula holds in the more general n-space $\mathbb{R}^n$...

Some Simple 3D Geometric Objects

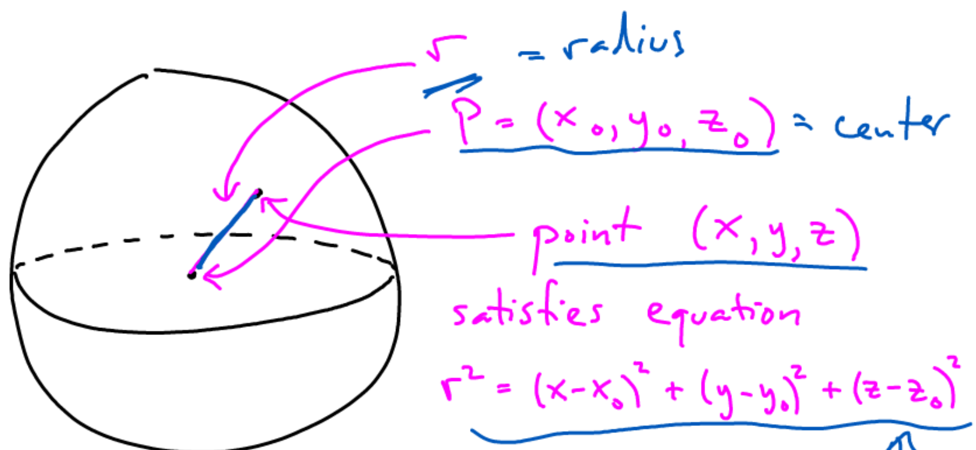
A <u>sphere</u> of radius $r>0$ centered at point $P=(x_0,y_0,z_0)$ is the set of all points in $\mathbb{R}^3$ at distance r from P. In other words, this sphere is the solution set to

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \leftarrow d((x,y,z),P)$$

i.e. the collection of points $(x,y,z) \in \mathbb{R}^3$ satisfying

$$\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2}$$

In particular, spheres are 3D versions of circles.



Ex: The sphere $(x-5)^2 + y^2 + (z+3)^2 = 5$ is centered at $P=(5,0,-3)$ and has radius $\sqrt{5}$. ☐

($y^2 = (y-0)^2$, $(z-(-3))$, $5 = (\sqrt{5})^2$)

NB: The equation $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$ describes only the <u>surface</u> of the sphere. A <u>solid ball</u> is described by $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \le r^2$.

<u>NB</u>: The object with the equation

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

is <u>NOT</u> a sphere in $\mathbb{R}^3$. In fact, this equation describes the surface of an <u>infinite circular cylinder</u> in $\mathbb{R}^3$... In $\mathbb{R}^2$ it describes a circle. The point: the geometric object described by an equation can depend on the "<u>ambient space</u>", i.e. $\mathbb{R}^3$ in the first example and $\mathbb{R}^2$ in the second.

We'll talk a lot more about surfaces soon...